* Lecture 3

Learning Objectives

- To apply the summation and product notation
- To define a matrix
- To solve problems on matrix summation, subtraction and multiplication
- To find the transpose of a matrix
- To calculate the inverse of a matrix

* 3.1 Summation Notation

* Defination 3.1
* $\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$

Example 3.1.1

$$\sum_{i=1}^{4} 2i = 2(1) + 2(2) + 2(3) + 2(4)$$
$$= 2 + 4 + 6 + 8$$
$$= 20$$

Summation Notation

* Example 3.1.2

$r = 1$ $\quad$ $r = 2$ $\quad$ $r = 3$ $\quad$ $r = 4$

$$\sum_{r=1}^{6} r(r+1) = 1(1+1) + 2(2+1) + 3(3+1) + 4(4+1)$$
$$+5(5+1) + 6(6+1)$$

$$= 2 + 6 + 12 + 20 + 30 + 42$$
$$= 112$$

* 3.2 Product Notation

* Defination 3.2
* $\prod_{i=m}^{n} a_i = (a_m)(a_{m+1}) \cdots (a_n)$

* Example 3.2.1

$$\prod_{i=1}^{4} 2i = 2(1) \cdot 2(2) \cdot 2(3) \cdot 2(4)$$
$$= 2 \cdot 4 \cdot 6 \cdot 8$$
$$= 384$$

Product Notation

- Example 3.2.2

$$\prod_{r=3}^{6} 2r - 1 = [2(3) - 1][2(4) - 1][2(5) - 1][2(6) - 1]$$
$$= (5)(7)(9)(11)$$
$$= 3465$$

* 3.3 Matrices

* Defination 3.3
* A matrix is a rectangular array of numbers arranged in rows and columns.
* A matrtix with m rows and n columns is called a matrix of order $m \times n$.

* 3.3 Matrices

* Example 3.3.1
* Sales amount (RM) of 3 restaurants over the last 3 months.

Months	New town Restaurant	Moonbucks	Teabeans
May	3293	5021	3129
June	3012	4892	2876
July	2312	5301	3421

* $M = \begin{bmatrix} 3293 & 5021 & 3129 \\ 3012 & 4892 & 2876 \\ 2312 & 5301 & 3421 \end{bmatrix}$
* The order of this matrix is 3×3.

* 3.3 Matrices

* A (1 x *n*) matrix is also called a row vector.

$$\begin{pmatrix} 1 & 0 & -3 \end{pmatrix}$$

* An (*m* x 1) matrix is also called a column vector

$$\begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

* 3.3 Matrices

* Entry of Matrix

Entry or element

$$A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 5 \end{pmatrix}$$

a_{ij} is the entry in the i-th row and j-th column.

* Example 3.3.2

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 5 \end{pmatrix}$$

$$a_{11} = 1, a_{12} = 0, a_{13} = -3$$

* 3.4 Matrix addition and subtraction

* If A = (a_{ij}) and B = (b_{ij}), where A and B have the same order, we get A + B by adding corresponding entries a_{ij} and b_{ij}

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 4 \\ -3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2 & 0+3 & -2+4 \\ 0+(-3) & 3+0 & 5+1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 3 & 2 \\ -3 & 3 & 6 \end{pmatrix}$$

* Subtraction works in a similar way

3.4 Matrix Multiplication

- Example 3.4.1

$$AB=\begin{pmatrix}-2 & 0 & 4 & 1\\ 1 & -3 & 0 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & -2\\ 0 & 1 & 3\\ 3 & -2 & 1\\ -1 & 2 & 0\end{pmatrix}$$

$$=\begin{pmatrix}-2(1)+0(0)+4(3)+1(-1) & -2(2)+0(1)+4(-2)+1(2) & -2(-2)+0(3)+4(1)+1(0)\\ 1(1)+(-3)(0)+0(3)+2(-1) & 1(2)+(-3)(1)+0(-2)+2(2) & 1(-2)+(-3)(3)+0(1)+2(0)\end{pmatrix}$$

$$=\begin{pmatrix}9 & -10 & 8\\ -1 & 3 & -11\end{pmatrix}$$

* 3.4 Matrix Multiplication

- An (n x m)-matrix can be multiplied together with an (m x k)-matrix to give an (n x k)-matrix.
  - The first two matrices are said to be compatible for multiplication, or to be "conformable"

$$A = \begin{pmatrix} -2 & 0 & 4 & 1 \\ 1 & -3 & 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 2 & 0 \end{pmatrix}$$

A has order 2 x 4

B has order 4 x 3

AB exists and has order 2 x 3.

3.5 Transpose

* From a matrix A we get its transpose A^{T} by interchanging the rows and columns
* We write down each row of A as a column of a new matrix, which becomes A^{T}

$$A = \begin{pmatrix} 2 & -3 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 2 & -2 \\ -3 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(A + B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$$

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$$

* 3.6 Identity and Zero Matrix

* Let $n \geq 1$, n an integer.
* The matrix

$$I_n = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

is the identity matrix of order n

* Let $n \geq 1$, n an integer.

$$O_n = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

is the zero matrix of order n

Sometimes we just write I or O instead of I_n or O_n

* It has 1 everywhere in the leading diagonal (top left to bottom right), and 0 everywhere else
* $IA = A$ and $AI = A$ whenever the product exists

* $O + A = A + O$
* $O \cdot A = O$ (= $A \cdot O$ possibly)

* 3.7 “Additive inverse” or Negative

* Given A, if we negate all the entries we get $-A$
* Then $A + (-A) = O = -A + A$

* 3.8 Multiplicative Inverse

* Given a square (same number of rows as columns) matrix A:
  * If there exists another square matrix A^{-1} with the same order as A and such that $AA^{-1} = I = A^{-1}A$
  * Then A^{-1} is called the inverse of A, and A is said to be invertible

$$A = \begin{pmatrix} 5 & -2 \\ 4 & -2 \end{pmatrix}, \mathrm{B} = \begin{pmatrix} 1 & -1 \\ 2 & -2.5 \end{pmatrix}$$

$$AB = \begin{pmatrix} 5 & -2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & -2.5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore B = A^{-1} \; or \; A = B^{-1}$$

Determinants

- Associated with any square matrix A is a special number called the determinant of A
- It has a role in providing info about the extent to which A operates on a space of vectors to cause an expansion or contraction of that space
- It is denoted by $| A |$ or det A or det(A)

* Determinant

* In the 2 x 2 case, a general matrix is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

* Then the determinant is | A |, and we write

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

* Multiplicative Inverse: Theorem

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

provided that $ad - bc \neq 0$.

* Partial Proof

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & -ab+ba \\ cd-dc & -bc+ad \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

* Multiplicative Inverse

* A matrix is called singular if its determinant is zero

* So we've seen that a square matrix is invertible if and only if it is nonsingular

Example 3.8.1

$$A = \begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} = 1 \cdot 4 - (-3) \cdot 2 = 10$$

$$A^{-1} = \frac{1}{10} \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$$